Potential Theory and Nonlinear Elliptic Equations
Lecture 1

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Abstract

We will use **potential theory** to study nonlinear elliptic equations and inequalities with measure coefficients and data of the type

$$-\Delta u = \sigma u^q + \mu, \quad q \in \mathbb{R} \backslash \{0\},$$

in a domain $\Omega \subseteq \mathbb{R}^n$, or on a weighted Riemannian manifold M, as well as more general equations

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = f(x, u, \nabla u) \quad \text{in } \Omega.$$

with certain quasilinear operators of p-Laplace type, and fully nonlinear operators of k-Hessian type in place of Δ.

Analogous problems for integral operators with positive kernels (in particular, Green's kernels) will be discussed. This includes sharp conditions for the existence of weak solutions, pointwise estimates of positive solutions, regularity and uniqueness results.

This minicourse is based on joint work with Michael Frazier, Alexander Grigor'yan and Yuhua Sun, and my former students Nguyen Cong Phuc, Ben Jaye, Dat Tien Cao, Stephen Quinn, Adisak Seesanea.

Publications 1

1. *Pointwise estimates of solutions to semilinear elliptic equations and inequalities*, ***J. D′Analyse Math.***, **137** (2019) 529–558 (with Alexander Grigor'yan)
2. *Pointwise estimates of solutions to nonlinear equations for nonlocal operators*, ***Ann. Scuola Norm. Super. Pisa***, **20** (2020) 721–750 (with Alexander Grigor'yan)
3. *Superlinear elliptic inequalities on manifolds*, ***J. Funct. Analysis***, **278** (2020), Art. 108444 (with Alexander Grigor'yan and Yuhua Sun)
4. *Quasilinear elliptic equations with sub-natural growth terms and nonlinear potential theory*, ***Atti Acad. Nat. Rend. Lincei*** **30** (2019), 733–758
5. *Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.*** (April, 2021), https://doi.org/10.1515/acv-2021-0004
6. *BMO solutions to quasilinear equations of p-Laplace type*, ***Ann. Inst. Fourier*** (May, 2021), arXiv:2105.05282 (with Nguyen Cong Phuc)

Publications 2

1. *Quasilinear and Hessian equations of Lane–Emden type*, ***Ann. Math.***, **168** (2008), 859–914 (with Nguyen Cong Phuc)
2. *Weighted norm inequalities of $(1, q)$-type for integral and fractional maximal operators*, Harmonic Analysis, Partial Differential Equations and Applications (2017), 217–238 (with Stephen Quinn)
3. *A sublinear version of Schur's lemma and elliptic PDE*, ***Analysis*** & ***PDE***, **11** (2018), 439–466 (with Stephen Quinn)
4. *Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.***, **13** (2020), 53–74 (with Adisak Seesanea)
5. *Pointwise estimates of Brezis–Kamin type for solutions of sublinear elliptic equations*, ***Nonlin. Analysis***, **146** (2016) 1–19 (with Dat Tien Cao)
6. *Nonlinear elliptic equations and intrinsic potentials of Wolff type*, ***J. Funct. Analysis***, **272** (2017) 112–165 (with Dat Tien Cao)

Additional literature

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2. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, Amer. Math.Soc./Intern. Press Studies in Adv. Math., **47**, 2009.
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6. X.-J. Wang, *The k-Hessian Equation*, Lecture Notes Math., **1977**, Springer, Berlin, 2009.

Introduction

We consider the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu \quad \text{in } \Omega, \tag{1}$$

where $\operatorname{div}\mathcal{A}$ is the so-called $\mathcal{A}$-Laplacian, and $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear measurable function subject to standard growth and monotonicity assumptions of order $p \in (1, \infty)$ discussed below.

A model example is $\mathcal{A}(\cdot, \xi) = \xi\,|\xi|^{p-2}$ $(\xi \in \mathbb{R}^n)$, which corresponds to the p-Laplace operator $\Delta_p u = \operatorname{div}(\nabla u\,|\nabla u|^{p-2})$. In the linear case $p = 2$ we have the classical Laplace operator $\Delta u = \operatorname{div}(\nabla u)$.

If $\mathcal{A}(x, \xi) = A(x)\xi$, where $A(x)$ $(x \in \Omega)$ is a linear matrix function, then the $\mathcal{A}$-Laplacian is a linear uniformly elliptic second-order differential operator $\mathcal{L}u = \operatorname{div}(A\nabla u)$ with bounded measurable coefficients. A related integral equation

$$u = \mathbf{G}(\sigma u^q) + \mathbf{G}\mu \quad \text{in } \Omega, \tag{2}$$

where $\mathbf{G} = (-\mathcal{L})^{-1}$ is Green's operator for $\mathcal{L}$, with positive Green's kernel G on $\Omega \times \Omega$, will be treated below [Grigor'yan-Verbitsky 2020].

Structural assumptions on $\mathcal{A}(x, \xi)$

Let $1 < p < \infty$, and let $\Omega \subseteq \mathbb{R}^n$ be an open set.
We assume that $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the following structural assumptions:

$$x \to \mathcal{A}(x, \xi) \quad \text{is measurable for all } \ \xi \in \mathbb{R}^n,$$

$$\xi \to \mathcal{A}(x, \xi) \quad \text{is continuous for a.e. } \ x \in \Omega,$$

and there are constants $0 < \alpha \leq \beta < \infty$, such that for a.e. x in Ω, and for all ξ in $\mathbb{R}^n$,

$$\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}, \tag{3}$$

$$(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0 \quad \text{if } \ \xi_1 \neq \xi_2. \tag{4}$$

If $p = 2$ and $\mathcal{A}(x, \xi) = A(x)\xi$ is linear, these are the usual uniform ellipticity and essential boundedness assumptions for $\mathcal{L}u = \mathrm{div}(A \nabla u)$.

$\mathcal{A}$-superharmonic solutions

More generally, consider the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = f(x, u, \nabla u) \quad \text{in } \Omega. \tag{5}$$

If $f \geq 0$, then the right-hand side of (5) is a nonlinear source term, and it is natural to consider $\mathcal{A}$-*superharmonic* solutions u to (5).
By $\mathcal{M}^+(\Omega)$ we denote the class of (locally finite) nonnegative *Radon measures* in Ω. A nonlinear potential theory for the equation with measure right-hand side $\mu \in \mathcal{M}^+(\Omega)$,

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu, \tag{6}$$

where u is $\mathcal{A}$-superharmonic, was developed by [Kilpeläinen-Malý 1994].
They obtained sharp bilateral pointwise estimates of positive solutions u to (6) in terms of Wolff potentials discussed below (more accurately they are called Havin-Maz'ya-Wolff potentials).

Special case: p-superharmonic functions

A function $u \in W^{1,p}_{\rm loc}(\Omega)$ is called p-harmonic if it satisfies the homogeneous equation $\Delta_p u = 0$. Every p-harmonic function has a continuous representative which coincides with u a.e.

Then p-superharmonic functions are defined via a *comparison principle*: A function $u\colon \Omega \to (-\infty, \infty]$ is said to be p-superharmonic if u is *lower semicontinuous*, not identically infinite in any component of Ω, and, whenever $D \subset\subset \Omega$ and $h \in C(\overline{D})$ is p-harmonic in D, then

$$h \le u \text{ on } \partial D \Longrightarrow h \le u \text{ in } D.$$

A p-superharmonic function $u \ge 0$ does not necessarily belong to $\mathrm{W}^{1,p}_{\rm loc}(\Omega)$, but its truncates $T_k(u) = \min(u, k) \in \mathrm{W}^{1,p}_{\rm loc}(\Omega)$, $\forall k > 0$. In addition, $T_k(u)$ are supersolutions, that is,

$$-\mathrm{div}(|\nabla T_k(u)|^{p-2}\nabla T_k(u)) \ge 0,$$

in the distributional sense. The **generalized gradient** Du of a p-superharmonic function $u \ge 0$ is defined by

$$Du = \lim_{k\to+\infty} \nabla(T_k(u)).$$

p-superharmonic functions

We note that every p-superharmonic function u has a *quasi-continuous* representative, which coincides with u quasi-everywhere (q.e.), i.e., everywhere except for a set of p-capacity zero. We will assume that u is always chosen this way.

Let u be p-superharmonic, and let $1 \le r < \frac{n}{n-1}$. Then $|Du|^{p-1}$, and consequently $|Du|^{p-2}Du$, belongs to $L^r_{\text{loc}}(\Omega)$. This allows us to define a **nonnegative distribution** $-\Delta_p u$ for each p-superharmonic function u by

$$-\langle \Delta_p u, \varphi \rangle = \int_\Omega |Du|^{p-2} Du \cdot \nabla \varphi \, dx, \tag{7}$$

for all $\varphi \in C_0^\infty(\Omega)$. Then by the Riesz representation theorem there exists a unique Radon measure $\mu = \mu(u) \in \mathcal{M}^+(\Omega)$ so that

$$-\Delta_p u = \mu \quad \text{in } \Omega. \tag{8}$$

These definitions are easy to generalize to $\mathcal{A}$-superharmonic functions.

Quasilinear, fully nonlinear, nonlocal operators

- For quasilinear equations and related inequalities of the type

$$-\Delta_p u = \sigma\, u^q + \mu,$$

  we distinguish between the cases:
  (a) $0 < q < p-1$ (sub-natural growth) [Cao-Verbitsky 2016/17], (b) $q = p-1$ (natural growth) [Jaye-Verbitsky 2012], [Jaye-Maz'ya-Verbitsky 2013], (c) $q > p-1$ (super-natural growth) [Phuc-Verbitsky 2006/08/21], (d) $q < 0$ (negative exponent) [Grigor'yan-Verbitsky 2019].
  More general operators $\mathrm{div}\mathcal{A}(x, \nabla u)$ in place of Δ_p.
- Fully nonlinear k-Hessian equations ($k = 1, \ldots, n$):

$$F_k[u] = \sigma\, |u|^q + \mu,$$

  in the class of k-convex functions u, $0 < q < k$ (**sub-natural growth**). $F_k[u]$ = the sum of the $k \times k$ principal minors of $D^2 u$.
- Nonlocal fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, integral equations with positive Green's kernels $G(x, y)$.

Extensions

Consider another model equation of the type (5), namely,

$$-\Delta_p u = \sigma \frac{|\nabla u|^q}{u^\gamma} + \mu \quad \text{in } \Omega. \tag{9}$$

Even for $\sigma = \text{const}$, this is a challenging problem. We will treat the special case with singular natural growth in the gradient ($q = p$, $\gamma = 1$) and constant $\sigma > 0$ [Cao-Verbitsky 2017].
In the case $p = 2$ and $q = 0$, $\gamma > 0$, sharp estimates of solutions u for σ changing sign, have been obtained by [Grigor'yan-Verbitsky 2019].
Another important special case is $\gamma = 0$. For $p = 2$ and $\sigma = \text{const}$, see [Hansson-Maz'ya-Verbitsky 1999], [Frazier-Verbitsky 2017/21]. For $p \neq 2$, $q = p$ (natural growth in the gradient) see [Jaye-Verbitsky 2012/13]. For $q \neq p$, $\sigma = \text{const}$, equations (9) are a subject of extensive studies [Nguyen Cong Phuc et al. 2015/20/21].
The potential theory approach is useful in studies of other important nonlinear equations and systems, as well as their analogues on **Riemannian manifolds** [Grigor'yan-Sun-Verbitsky 2020].

Riesz potentials

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. By $\mathbf{I}_\alpha \mu$ $(0 < \alpha < n)$, we denote the Riesz potential of order α, defined by

$$\mathbf{I}_\alpha \mu(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} d\mu(y), \quad x \in \mathbb{R}^n. \tag{10}$$

A necessary and sufficient condition for $\mathbf{I}_\alpha \mu \not\equiv +\infty$ is given by

$$\int_{\mathbb{R}^n} \frac{1}{(|y|+1)^{n-\alpha}} d\mu(y) < +\infty.$$

In this case, we actually have $\mathbf{I}_\alpha \mu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, so that $\mathbf{I}_\alpha \mu < +\infty$ dx-a.e.
If $d\mu = f\, dx$, where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we write $\mathbf{I}_\alpha f$ in place of $\mathbf{I}_\alpha (f dx)$.
The operator $\mathbf{I}_\alpha$ is a fractional integral of order α. This is usually expressed in a symbolic form, justified using Fourier transforms,

$$\mathbf{I}_\alpha = c\,(-\Delta)^{-\frac{\alpha}{2}},$$

where $c = c(\alpha, n) > 0$ is a normalization constant.

Wolff potentials

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. Let $0 < \alpha < n$ and $1 < r < \infty$. The nonlinear Wolff potential $\mathbf{W}_{\alpha,r}\mu$ is defined by

$$\mathbf{W}_{\alpha,r}\mu(x) := \int_0^\infty \left(\frac{\mu(B(x,\rho))}{\rho^{n-\alpha r}}\right)^{\frac{1}{r-1}} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n. \tag{11}$$

Here $B(x,\rho)$ is a Euclidean ball of radius ρ centered at $x \in \mathbb{R}^n$. Notice that $\mathbf{W}_{\alpha,r}\mu \not\equiv +\infty$ if and only if

$$\int_1^\infty \left(\frac{\mu(B(0,\rho))}{\rho^{n-\alpha r}}\right)^{\frac{1}{r-1}} \frac{d\rho}{\rho} < +\infty. \tag{12}$$

In the special case $\alpha = 1$, $r = p$, Wolff potentials $\mathbf{W}_{1,p}$ play an important role in the theory of quasi-linear equations [Kilpeläinen-Malý 1994]. Another special case $\alpha = \frac{k}{k+1}$, $r = k$ ($k = 1, \ldots, [\frac{n}{2}]$) is important for k-Hessian equations [Labutin 2003], [Trudinger-Wang 2002].

Riesz potentials and Wolff potentials

Remark. If $r = 2$ and $0 < \alpha < \frac{n}{2}$, we have

$$\mathbf{W}_{\alpha,2}\mu(x) = \frac{1}{n-2\alpha}\, \mathbf{I}_{2\alpha}\mu(x), \quad x \in \mathbb{R}^n.$$

Proof.

For any $x \in \mathbb{R}^n$, by Fubini's theorem,

$$\begin{aligned} \mathbf{W}_{\alpha,2}\mu(x) &= \int_0^\infty \frac{\mu(B(x,\rho))}{\rho^{n-2\alpha}} \frac{d\rho}{\rho} \\ &= \int_0^\infty \int_{B(x,\rho)} d\mu(y) \frac{d\rho}{\rho^{n-2\alpha+1}} \\ &= \int_{\mathbb{R}^n} \int_{|x-y|}^\infty \frac{d\rho}{\rho^{n-2\alpha+1}} d\mu(y) \\ &= \frac{1}{n-2\alpha} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-2\alpha}} = \frac{1}{n-2\alpha}\, \mathbf{I}_{2\alpha}\mu(x). \end{aligned}$$

□

The Kilpeläinen-Malý theorem

Theorem (Kilpeläinen-Malý 1994)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ *and* $1 < p < n$. *Then there exists a constant* $C = C(p, n)$ *such that*

$$C^{-1}\mathbf{W}_{1,p}\mu(x) \le u(x) \le C\,\mathbf{W}_{1,p}\mu(x), \qquad x \in \mathbb{R}^n, \tag{13}$$

for any p*-superharmonic solution of the equation*

$$-\Delta_p u = \mu \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0. \tag{14}$$

In the case $p \ge n$ *there are no nontrivial solutions to* (14) *on* $\mathbb{R}^n$.

Remark. This fundamental theorem, along with its important *local version*, holds for more general equations $-\mathcal{L}u = \mu$ with $\mathcal{A}$-Laplace operators $\mathcal{L} = \mathrm{div}\mathcal{A}(x, \nabla u)$ in place of Δ_p, under conditions (3), (4) (due to the same authors). In other words, $(-\mathcal{L})^{-1}\mu \approx \mathbf{W}_{1,p}\mu$.

Riesz capacities and p-capacity

In the potential theory related to quasilinear equations of p-Laplace type, the p-capacity plays a fundamental role. It is defined, for a **compact** set $E \subset \Omega$ by

$$\mathrm{Cap}_p(E, \Omega) = \inf \left\{ \|\nabla u\|^p_{L^p(\Omega)} \colon u \geq 1 \text{ on } E, \ u \in C_0^\infty(\Omega) \right\}. \qquad (15)$$

For any set $E \subset \mathbb{R}^n$, the Riesz capacity of order (α, r) is defined by

$$\mathrm{Cap}_{\alpha, r}(E) = \inf \left\{ \|f\|^r_{L^r(\mathbb{R}^n)} \colon \mathrm{I}_\alpha f \geq 1 \text{ on } E, \ f \geq 0 \right\}. \qquad (16)$$

It is easy to see that, for $\Omega = \mathbb{R}^n$, and $\alpha = 1$, $r = p$, the Riesz capacity is equivalent to the p-capacity,

$$\mathrm{Cap}_{1, p}(E) \approx \mathrm{Cap}_p(E), \quad E \subset \mathbb{R}^n \text{ compact},$$

where the equivalence constants depend only on p and n.

Capacities associated with Sobolev spaces

In our study of equations (1), we also use related capacities associated with Sobolev spaces $W^{\alpha,r}(\Omega)$, where α, r may depend on p, q [Phuc-Verbitsky 2008, 2009]. Here
The capacity associated with $W^{\alpha,r}(\Omega)$ is defined by

$$\mathrm{cap}_{\alpha,r}(E,\Omega) = \inf \left\{ \|u\|^r_{W^{\alpha,r}(\mathbb{R}^n)} \colon u \geq 1 \text{ on } E, \ u \in C_0^\infty(\Omega) \right\}, \quad (17)$$

for compact sets $E \subset \Omega$.
Here

$$\|u\|_{W^{\alpha,r}(\mathbb{R}^n)} = \|u\|_{L^r(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^r(\mathbb{R}^n)}.$$

By $W_0^{\alpha,r}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ functions with respect to the norm $\|u\|_{W^{\alpha,r}}$.

Capacities associated with homogeneous Sobolev spaces

Similarly, we define capacities associated with homogeneous Sobolev spaces $\dot{W}^{\alpha,r}(\Omega)$ ($0 < \alpha < n$ and $1 < r < \infty$),

$$\mathrm{Cap}_{\alpha,r}(E,\Omega) = \inf \left\{ \|u\|^r_{\dot{W}^{\alpha,r}(\mathbb{R}^n)} : u \ge 1 \text{ on } E,\ u \in C_0^\infty(\Omega) \right\}, \quad (18)$$

for compact sets $E \subset \Omega$. Here

$$\|u\|_{\dot{W}^{\alpha,r}(\mathbb{R}^n)} = \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^r(\mathbb{R}^n)}.$$

By $\dot{W}_0^{\alpha,r}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ functions with respect to the norm $\|u\|_{\dot{W}^{\alpha,r}}$. In the case $\Omega = \mathbb{R}^n$, these capacities are equivalent to Riesz capacities,

$$\mathrm{Cap}_{\alpha,r}(E,\mathbb{R}^n) \approx \mathrm{Cap}_{\alpha,r}(E), \quad E \subset \mathbb{R}^n \text{ compact}.$$

Moreover, in the special case $\alpha = 1$, $r = p$, the p-capacity

$$\mathrm{Cap}_p(E) \approx \mathrm{Cap}_{1,p}(E).$$

Example: A sublinear problem on $\Omega = (0, \infty)$

One-dimensional case

For $0 < q < 1$, consider the *homogeneous* problem

$$-u'' = \sigma\, u^q, \quad u \geq 0, \quad \text{in } \Omega = (0, \infty). \tag{19}$$

Here $\sigma \in \mathcal{M}^+(\Omega)$, u is a **concave** ("superharmonic") function. The Green function $G(x, y) = \min(x, y)$, and the Green potential

$$\mathbf{G}\sigma(x) = \int_0^x y\, d\sigma(y) + x \int_x^\infty d\sigma(y), \quad x > 0.$$

A general solution u to (19) satisfies the integral equation

$$u(x) = \int_0^x y\, u^q(y)\, d\sigma(y) + x \int_x^\infty u^q(y)\, d\sigma(y) + ax + b. \tag{20}$$

We assume $u(0) = 0$, $\lim_{x\to\infty} \frac{u(x)}{x} = 0$, so that $a = b = 0$, and

$$u = \mathbf{G}(u^q d\sigma).$$

Notice that $u'(x) = \int_x^\infty u^q(y) d\sigma(y) \geq 0$, so u is non-decreasing.

Example: A sublinear problem on $\Omega = (0, \infty)$

One-dimensional case

Theorem 1 (Quinn-Verbitsky 2017)

Let $0 < q < 1$ and $\sigma \geq 0$. Any nontrivial solution to (20) *satisfies*

$$C^{-1}\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathrm{K}\sigma(x)\Big] \tag{21}$$

$$\leq u(x) \leq C\,\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathrm{K}\sigma(x)\Big], \tag{22}$$

where $\mathrm{K}\sigma(x) = x\Big(\int_x^{+\infty} y^q d\sigma(y)\Big)^{\frac{1}{1-q}}$, *and* $C = C(q)$.

A nontrivial solution exists if and only if, for some (or all) $a > 0$,

$$\int_0^a y\, d\sigma(y) + \int_a^{+\infty} y^q d\sigma(y) < +\infty. \tag{23}$$

Remarks

One-dimensional case

1. We actually have $u \approx (\mathbf{G}\sigma)^{\frac{1}{1-q}} + \mathbf{K}\sigma$ for any nontrivial solution u.
2. The lower bound holds for any nontrivial supersolution $u \geq \mathbf{G}(u^q d\sigma)$, and the upper bound for any subsolution $0 < u \leq \mathbf{G}(u^q d\sigma)$.
3. The lower pointwise bound $u \geq (1-q)^{\frac{1}{1-q}} (\mathbf{G}\sigma)^{\frac{1}{1-q}}$ is known for a general kernel G which satisfies the strong maximum principle [Grigor'yan-Verbitsky 2019]; the constant $(1-q)^{\frac{1}{1-q}}$ is sharp. The term $\mathbf{K}\sigma$ strengthens the lower estimate, matches the upper estimate.
4. Similar pointwise estimates hold for $\Omega = (0, 1)$ with Green's function $G(x, y) = \min[x(1-y), y(1-x)]$. Existence criteria in one dimension are due to [Yong Zhang, 1994], [Zhong-li Wei, Shao-zhu Chen, 2005].
5. For the corresponding *inhomogeneous* problem with measures μ, σ

$$u = \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu, \quad u \geq 0 \quad \text{in } \Omega = (0, \infty), \tag{24}$$

we have $u \approx (\mathbf{G}\sigma)^{\frac{1}{1-q}} + \mathbf{K}\sigma + \mathbf{G}\mu$ [Verbitsky 2021].

Weighted norm inequalities on $\Omega = (0, \infty)$

One-dimensional case

Pointwise estimates in Theorem 1 are based on the following theorem.

Let $\Omega = (0, \infty)$ and $0 < q < 1$. Consider the $(1, q)$-weighted norm inequality [Quinn-Verbitsky 2017] (for a finite constant $\varkappa$),

$$\|u\|_{L^q(\Omega,\sigma)} \leq \varkappa \int_0^\infty |u''| \, dx, \tag{25}$$

for all concave functions u in Ω with $u(0) = 0$, $\lim_{x\to\infty} \frac{u(x)}{x} = 0$.

Equivalently, in terms of Green's potentials $u = \mathbf{G}\nu$,

$$\|\mathbf{G}\nu\|_{L^q(\Omega,\sigma)} \leq \varkappa \|\nu\|, \tag{26}$$

for all measures $\nu \in \mathcal{M}^+(\Omega)$, where $\|\nu\| = \nu(\Omega)$.

Weighted norm inequalities on $\Omega = (0, \infty)$

(continuation)

Theorem 2 (Quinn-Verbitsky 2017)

Let $0 < q < 1$ *and* $\sigma \geq 0$.

(i) The best constant $\varkappa$ *in* (25) *or* (26) *satisfies*

$$\varkappa = \left(\int_0^{+\infty} x^q d\sigma(x) \right)^{\frac{1}{q}}. \tag{27}$$

(ii) There exists a nontrivial solution $u = \mathbf{G}(u^q d\sigma)$ *to* (20) *such that* $u'(0) < \infty$ *if and only if* $\varkappa < \infty$.

Notice that we impose an additional assumption on the solution u:

$$u'(0) = \|u''\| = \int_0^\infty u^q d\sigma < \infty. \tag{28}$$

General integral and differential equations

Our goal is to study pointwise behavior of positive solutions to nonlinear integral equations (and related inequalities) of the type

$$u(x) = \int_\Omega G(x,y)\, g(u(y)) d\sigma(y) + h,$$

where (Ω, σ) is a locally compact measure space,
$G(x,y)\colon \Omega \times \Omega \to (0, +\infty]$ a kernel, $g\colon [0,\infty) \to [0,\infty)$ a monotone function, $h \geq 0$ a measurable function.
Motivation: singular solutions to semilinear (fractional) Laplace problems

$$(-\Delta)^{\frac{\alpha}{2}} u = g(u)\sigma + \mu \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

with measure coefficients σ, μ; $g(u) = u^q$, $q \in \mathbb{R} \setminus \{0\}$; $0 < \alpha < n$. Domains $\Omega \subseteq \mathbb{R}^n$, or Riemannian manifolds, with positive Green's function G. In some cases: necessary and sufficient conditions for existence of positive solutions. ***Generalizations*** to quasilinear and Hessian equations.

Local case: weighted manifolds

(joint work with Alexander Grigor'yan)

Let M be a Riemannian manifold, and let ω be a smooth positive function on M. Consider the measure m on M given by $dm = \omega dm_0$, where m_0 is the Riemannian measure of M. The couple (M, m) is called a weighted manifold. Set

$$\mathrm{div}_\omega = \frac{1}{\omega} \circ \mathrm{div} \circ \omega.$$

Here div and ∇ are the divergence and the gradient operators of the Riemannian structure of M discussed below.

The (weighted) Laplace operator $\mathcal{L} = \Delta$ on (M, m) is defined by

$$\Delta = \mathrm{div}_\omega \circ \nabla$$

If $\omega = 1$ then Δ is the Laplace-Beltrami operator denoted by $\mathcal{L}_0$.

Weighted manifolds

(continuation)

The Laplace operator $\boldsymbol{\Delta}$ on $(\boldsymbol{M}, \boldsymbol{m})$ satisfies the same Product Rule and Chain Rule as the classical Laplace operator.

1. For $\boldsymbol{C}^2$ functions $\boldsymbol{u}, \boldsymbol{v}$ on $\boldsymbol{M}$,

$$\boldsymbol{\Delta}\,(\boldsymbol{uv}) = \boldsymbol{u}\,\boldsymbol{\Delta v} + 2\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla v}\rangle + \boldsymbol{v}\,\boldsymbol{\Delta u}. \tag{29}$$

Here $\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla v}\rangle$ is the inner product of the Riemannian gradients, which is independent of the weight $\boldsymbol{\omega}$.

2. For any $\boldsymbol{C}^2$ function ϕ defined on $\boldsymbol{u}\,(\boldsymbol{M})$,

$$\boldsymbol{\Delta}\phi\,(\boldsymbol{u}) = \phi'\,(\boldsymbol{u})\,\boldsymbol{\Delta u} + \phi''\,(\boldsymbol{u})\,|\boldsymbol{\nabla u}|^2\,. \tag{30}$$

The Laplace-Beltrami operator

If $\omega = 1$, the Laplace-Beltrami operator $\mathcal{L}_0 = \Delta$ acts on C^2 functions u on M. It is given in any chart $x_1, \dots, x_n$ by the formula

$$\mathcal{L}_0 u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(\sqrt{\det g}\, g^{ij} \partial_{x_j} u \right)$$

where g is the Riemannian metric, $\det g$ is the determinant of the matrix $g = (g_{ij})$, and (g^{ij}) is the inverse matrix of (g_{ij}).
The Riemannian measure m_0 is given in the same chart by

$$dm_0 = \sqrt{\det g}\, dx_1 \dots dx_n.$$

Notice that $\mathcal{L}_0$ is symmetric with respect to m_0.

The Laplace-Beltrami operator
(continuation)

The gradient operator ∇ is defined by

$$(\nabla u)^i = \sum_{j=1}^{n} g^{ij} \partial_{x_j} u. \tag{31}$$

The divergence $\mathbf{div}$ on vector fields F^i is defined by

$$\mathrm{div} F = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \partial_{x_i} \left(\sqrt{\det g}\, F^i \right). \tag{32}$$

In other words, $\mathcal{L}_0$ is represented in the form

$$\mathcal{L}_0 = \mathrm{div} \circ \nabla.$$

The weighted Laplace operator

Let (M, m) be a weighted manifold with $dm = \omega\, dm_0$.

Recall that ∇ is the Riemannian gradient and does not depend on the weight ω.

The (weighted) Laplace operator is defined by $\boldsymbol{\Delta} = \mathbf{div}_\omega \circ \nabla$.

From the definitions of ∇ and $\mathbf{div}$, see (31) and (32), it follows that

$$\Delta u = \frac{1}{\omega} \operatorname{div}(\omega \nabla u) = \frac{1}{\omega\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i}\left(\omega \sqrt{\det g}\, g^{ij} \partial_{x_j} u\right), \quad (33)$$

acting on C^2 functions u on M.