Potential Theory and Nonlinear Elliptic Equations
Lecture 5

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Publications

1. I. Verbitsky, *Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.*** (published online, April 2021), DOI: 10.1515/acv-2021-0004
2. Nguyen Cong Phuc and I. Verbitsky, BMO *solutions to quasilinear equations of* $\boldsymbol{p}$*-Laplace type*, ***Ann. Inst. Fourier*** (2021, to appear), arXiv:2105.05282
3. A. Seesanea and I. Verbitsky, *Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.***, **13** (2020) 53–74.
4. S. Quinn and I. Verbitsky, *A sublinear version of Schur's lemma and elliptic PDE*, ***Analysis & PDE***, **11** (2018) 439–466.
5. Dat Tien Cao and I. Verbitsky, *Nonlinear elliptic equations and intrinsic potentials of Wolff type*, ***J. Funct. Analysis***, **272** (2017) 112–165.
6. Dat Tien Cao and I. Verbitsky, *Pointwise estimates of Brezis–Kamin type for solutions of sublinear elliptic equations*, ***Nonlin. Analysis***, **146** (2016) 1–19.

Additional literature

1. D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der math. Wissenschaften, **314**, Springer, Berlin, 1996.
2. T. Kilpeläinen, T. Kuusi and A. Tuhola-Kujanpää, *Superharmonic functions are locally renormalized solutions*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, **28** (2011), 775–795.
3. T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137–161. Ann. Inst. H. Poincaré, Anal. Non Linéaire, **28** (2011), 775–795.
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5. V. G. Maz'ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, 2nd revised augm. ed., Grundlehren der math. Wissenschaften, **342**, Springer, Berlin, 2011.
6. X.-J. Wang, *The k-Hessian Equation*, Lecture Notes Math., **1977**, Springer, Berlin, 2009.

Finite energy solutions in the Sobolev space $\dot{W}_0^{1,2}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, and let $\mu, \sigma \in \mathcal{M}^+(\Omega)$. Let $0 < q < 1$.

Definition. There exists a positive $\dot{W}_0^{1,2}$-solution u (called finite energy solution) to the Dirichlet problem:

$$\begin{cases} -\Delta u = \sigma\, u^q + \mu & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1}$$

if $u \in \dot{W}_0^{1,2}(\Omega) \cap L^q_{loc}(\Omega, d\sigma)$, $u \geq 0$, and

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega \phi\, u^q \, d\sigma + \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_0^\infty(\Omega). \tag{2}$$

Here $\dot{W}_0^{1,2}(\Omega)$ is the homogeneous Sobolev (Dirichlet) space, that is, the closure of $C_0^\infty(\Omega)$ in the norm $||\nabla u||_{L^2(\Omega)}$.

Existence and uniqueness of finite energy solutions

Theorem 14 (Seesanea-Verbitsky 2020)

Let $\mathbf{0} < \boldsymbol{q} < \mathbf{1}$, $\boldsymbol{\Omega} \subseteq \mathbb{R}^{\boldsymbol{n}}$ Green domain. There exists a solution $\boldsymbol{u} \in \dot{\boldsymbol{W}}_0^{1,2}(\boldsymbol{\Omega})$ to the equation $-\boldsymbol{\Delta u} = \boldsymbol{\sigma u^q} + \boldsymbol{\mu}$ if and only if

$$\int_{\Omega} (\mathbf{G}\sigma)^{\frac{1+q}{1-q}} \, d\sigma < +\infty, \quad \int_{\Omega} (\mathbf{G}\mu) \, d\mu < +\infty. \tag{3}$$

Moreover, such a solution $\boldsymbol{u} \in \boldsymbol{L}^{1+q}(\boldsymbol{\Omega}, \boldsymbol{\sigma})$, and is unique.

In the special case $\boldsymbol{\Omega} = \mathbb{R}^{\boldsymbol{n}}$ $(\boldsymbol{n} \geq \mathbf{3})$, conditions (3) become

$$\int_{\mathbb{R}^n} (\mathbf{I}_2\sigma)^{\frac{1+q}{1-q}} \, d\sigma < +\infty, \quad \int_{\mathbb{R}^n} (\mathbf{I}_2\mu) \, d\mu < +\infty, \tag{4}$$

where $\mathbf{I}_2\boldsymbol{\sigma} = |\cdot|^{2-\boldsymbol{n}} \star \boldsymbol{\sigma}$ is the Newtonian potential of $\boldsymbol{\sigma}$.

A crucial integral inequality: finite energy solutions

It turns out that this problem is closely related to the **trace inequality** (in the **non-classical** case $1 + q < 2$):

$$\left(\int_\Omega |\phi|^{1+q}\, d\sigma\right)^{\frac{1}{1+q}} \le C\, \|\nabla \phi\|_{L^2(\Omega, dx)}, \quad \forall \phi \in C_0^\infty(\Omega).$$

A capacitary characterization [Mazy'a-Netrusov 1995]:

$$\int_0^{\sigma(\Omega)} \left(\frac{t}{\lambda(\sigma, t)}\right)^{\frac{1+q}{1-q}} dt < +\infty,$$

$\lambda(\sigma, t) = \inf\{\, \mathrm{cap}(E) : \ \sigma(E) \ge t\}$; equivalent to the Green potential condition [Seesanea-V. 2020] (for $\Omega = \mathbb{R}^n$ [Cascante-Ortega-V. 2000]):

$$\int_\Omega (\mathbf{G}\sigma)^{\frac{1+q}{1-q}}\, d\sigma < +\infty.$$

General solutions

We now focus on general (very weak) solutions to homogeneous ($\mu = 0$) equations (1) for $0 < q < 1$. For a bounded $\Omega \subset \mathbb{R}^n$ with C^2 boundary:

$$\int_\Omega -u\, \Delta\phi\, dx = \int_\Omega u^q\, \phi\, d\sigma, \quad \forall \phi \in C_0^2(\overline{\Omega}), \tag{5}$$

where $u \in L^1(\Omega, dx) \cap L^q(\Omega, \delta_\Omega\, d\sigma)$, is a non-trivial positive solution ($0 < u < +\infty$ $d\sigma$-a.e.) Similar definitions are known for Lipschitz Ω.

For bounded C^2 domains, Definition (5) is equivalent to:

$$u(x) = \int_\Omega G^\Omega(x,y)\, u^q(y)\, d\sigma(y), \quad x \in \Omega. \tag{6}$$

For arbitrary Green domains $\Omega \subseteq \mathbb{R}^n$: we use Definition (6).

Remark. We can treat non-homogeneous equations (1) with $\mu \neq 0$ in ***uniform*** domains Ω in a similar way: $u = \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu$.

Bounded solutions on $\mathbb{R}^n$

In the case $\Omega = \mathbb{R}^n$, $n \geq 3$: $G^{\mathbb{R}^n}(x, y) = c_n |x - y|^{2-n}$, and

$$u = \mathrm{I}_2(u^q \, d\sigma) \quad \text{in } \mathbb{R}^n. \tag{7}$$

Let $U(x) := \mathrm{I}_2\sigma(x)$ denote the Newtonian potential of $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$.

Theorem (Brezis-Kamin 1992)

Let $0 < q < 1$, $\sigma \in L^\infty_{loc}(\mathbb{R}^n)$ $(\sigma \neq 0)$. There exists a nontrivial bounded solution to equation (7) in $\mathbb{R}^n$ such that $\liminf_{|x| \to +\infty} u(x) = 0$ if and only if $U \in L^\infty(\mathrm{R}^n)$. Moreover, such a solution is unique, and satisfies the global estimates:

$$U(x)^{\frac{1}{1-q}} \leq u(x) \leq C \, U(x), \quad x \in \mathbb{R}^n. \tag{8}$$

Both the lower and the upper estimates in (8) are sharp in a sense.

Remark. More precise bilateral estimates use new ***nonlinear potentials***.

Extension of the Brezis-Kamin theorem

Homogeneous equations on $\Omega = \mathbb{R}^n$

Theorem 15 (Cao-Verbitsky 2016)

Let $0 < q < 1$ and $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ ($\sigma \neq 0$). Suppose for a constant C,

$$\sigma(F) \leq C \operatorname{cap}(F), \quad \forall \text{ compact sets } F \subset \mathbf{R}^n. \tag{9}$$

Then there exists a nontrivial solution $u > 0$ to (7) such that $\liminf_{|x|\to+\infty} u(x) = 0$, and for any solution u,

$$U(x)^{\frac{1}{1-q}} \leq u(x) \leq C\left(U(x) + U(x)^{\frac{1}{1-q}}\right), \quad x \in \mathbb{R}^n, \tag{10}$$

provided $U \not\equiv +\infty$ (otherwise there is no solution).

Remarks. 1. Both estimates are sharp as in the Brezis-Kamin theorem. **2.** The lower estimate holds for any $\sigma \geq 0$, without (9). **3.** Condition (9) is weaker than $U \in L^\infty(\mathbf{R}^n)$, and allows unbounded solutions u.

Weak solutions on $\mathbb{R}^n$: the radial case

In the radial case σ depends only on $r = |x|$ in $\mathbb{R}^n$, $n \geq 3$, and

$$U(r) = c_n \left(\frac{1}{r^{n-2}} \int_0^r t^{n-1}\, d\sigma(t) + \int_r^\infty t\, d\sigma(t) \right).$$

Theorem 16 (Cao-Verbitsky 2016)

Let $0 < q < 1$. Suppose σ is radial ($\sigma \neq 0$). Then (7) has a nontrivial (radial) solution iff

$$\int_0^1 \frac{t^{n-1} d\sigma(t)}{t^{(n-2)q}} < +\infty, \quad \text{and} \quad \int_1^{+\infty} t\, d\sigma(t) < +\infty.$$

Moreover, any solution u satisfies:

$$u(x) \approx U(r)^{\frac{1}{1-q}} + \frac{1}{r^{n-2}} \left(\int_0^r \frac{t^{n-1} d\sigma(t)}{t^{(n-2)q}} \right)^{\frac{1}{1-q}}.$$

Weak solutions on $\mathbb{R}^n$: a crucial weighted norm inequality

The problem of the existence of weak solutions to (7) is closely related to the following integral $(1, q)$-inequality in the case $0 < q < 1$: for all $\phi \in C_0^2(\mathbb{R}^n)$ such that $\phi \geq 0$, $\Delta\phi \leq 0$,

$$\left(\int_{\mathbb{R}^n} \phi^q \, d\sigma\right)^{\frac{1}{q}} \leq \varkappa \int_{\mathbb{R}^n} |\Delta\phi| \, dx.$$

Equivalently, a weighted norm inequality for Newtonian potentials holds:

$$\left(\int_{\mathbb{R}^n} (\mathbf{I}_2\nu)^q \, d\sigma\right)^{\frac{1}{q}} \leq \varkappa \, \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{11}$$

More generally, for the equation $(-\Delta)^{\frac{\alpha}{2}} u = \sigma\, u^q$, $0 < \alpha < n$,

$$\left(\int_{\mathbb{R}^n} (\mathbf{I}_\alpha\nu)^q \, d\sigma\right)^{\frac{1}{q}} \leq \varkappa \, \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n).$$

By $\varkappa$ we will denote the least constant in these inequalities.

Localized integral inequality

We will need a local version of the preceding inequality, where the measure $\sigma = \sigma_B$ is restricted to a ball B in $\mathbb{R}^n$:

$$\left(\int_B (\mathbf{I}_\alpha \nu)^q \, d\sigma \right)^{\frac{1}{q}} \le \varkappa_B \, \nu(\mathbb{R}^n), \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n).$$

The least constants $\varkappa_B$, where $B = B(x, r)$, are used to define a new **intrinsic** potential $\mathbf{K} = \mathbf{K}_\alpha$ of Wolff type,

$$\mathbf{K}\sigma(x) = \int_0^{+\infty} \frac{\left(\varkappa_{B(x,r)}\right)^{\frac{q}{1-q}}}{r^{n-\alpha}} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \tag{12}$$

Main Theorem

Theorem 17 (Cao-Verbitsky 2017)

Suppose $\Omega = \mathbb{R}^n$, *and* $0 < q < 1$. *Then* (7) *has a nontrivial (super) solution* u *such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if the following condition holds:*

$$\int_1^{+\infty} \frac{\sigma(B(0,r))}{r^{n-2}} \frac{dr}{r} + \int_1^{+\infty} \frac{\left(\varkappa_{B(0,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r} < +\infty. \tag{13}$$

Moreover, any solution u *to* (7) *satisfies*

$$u(x) \approx \left(\mathbf{I}_2\sigma(x)\right)^{\frac{1}{1-q}} + \int_0^{+\infty} \frac{\left(\varkappa_{B(x,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r}. \tag{14}$$

Remarks. 1. The second term is the intrinsic nonlinear potential $\mathbf{K}\sigma(x)$ defined by (12) with $\alpha = 2$. **2.** The upper estimate in (14) is proved only for the *minimal* solution in [Cao-V. 2017]; for all solutions in [V. 2021].

Existence of $W^{1,2}_{\mathrm{loc}}$ solutions (Sobolev regularity)

For the existence of a solution $\boldsymbol{u} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, an additional local version of the condition for finite energy solutions (Theorem 14) is needed:

$$\int_{B(0,R)} \left(\mathbf{I}_2 \sigma_{B(0,R)}\right)^{\frac{1+q}{1-q}} d\sigma < \infty, \quad \forall R > 0. \tag{15}$$

Theorem 18 (Cao-Verbitsky 2017)

Under the assumptions of the previous theorem, there exists a nontrivial weak (super) solution $\boldsymbol{u} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ *such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if* (15) *holds together with*

$$\int_1^{+\infty} \frac{\sigma(B(0,r))}{r^{n-2}} \frac{dr}{r} + \int_1^{+\infty} \frac{\left(\varkappa_{B(0,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r} < +\infty.$$

Moreover, global pointwise estimates (14) *hold.*

Wolff potentials

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. Let $0 < \alpha < n$ and $1 < p < \infty$.
The Wolff potential $\mathbf{W}_{\alpha,p}\mu$ (more accurately, the Havin-Maz'ya-Wolff potential) is defined by

$$\mathbf{W}_{\alpha,p}\mu(x) := \int_0^\infty \left(\frac{\mu(B(x,\rho))}{\rho^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n. \tag{16}$$

Recall that in the linear case $p = 2$ we have $\mathbf{W}_{\alpha,2}\mu = \mathbf{I}_{2\alpha}\mu$.

As we will prove below, $\mathbf{W}_{\alpha,p}\mu \not\equiv +\infty$ if and only if for $0 < \alpha < \frac{n}{p}$

$$\int_1^\infty \left(\frac{\mu(B(0,\rho))}{\rho^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty. \tag{17}$$

Remarks. 1. In the special case $\alpha = 1$, Wolff potentials $\mathbf{W}_{1,p}$ play an important role in the theory of quasilinear equations of p-Laplace type.
2. For $1 < p < 2 - \frac{\alpha}{n}$, we may have $\mathbf{W}_{\alpha,p}\mu \notin L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$.

Wolff potential estimates

We start with some useful estimates for Wolff potentials.

Lemma (Wolff potential estimates)

Suppose $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Let* $s = \min(1, p-1)$. *Then there exists a positive constant* $c = c(n, p, \alpha)$ *such that, for all* $x \in \mathbb{R}^n$ *and* $R > 0$,

$$\begin{aligned} c^{-1} \int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} &\leq \inf_{B(x,R)} \mathrm{W}_{\alpha,p}\sigma \\ &\leq \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} [\mathrm{W}_{\alpha,p}\sigma(y)]^s \, dy \right)^{\frac{1}{s}} \\ &\leq c \int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned} \tag{18}$$

Wolff potential estimates

(continuation)

Proof: WLOG assume $x = 0$. We first prove the last estimate in (18). Clearly,

$$\frac{1}{|B(0,R)|}\int_{B(0,R)} [W_{\alpha,p}\sigma(y)]^s \, dy \le I_1 + I_2,$$

where

$$I_1 = \frac{1}{|B(0,R)|}\int_{B(0,R)} \left(\int_0^R \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dy,$$

$$I_2 = \frac{1}{|B(0,R)|}\int_{B(0,R)} \left(\int_R^\infty \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dy.$$

To estimate I_2, notice that since $B(y,r) \subset B(0,2r)$ for $y \in B(0,R)$ and $r > R$, it follows

$$I_2 \le \left(\int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s.$$

Wolff potential estimates

(continuation)

To estimate I_1, suppose first that $p \geq 2$ so that $s = 1$. Then using Fubini's theorem and Jensen's inequality we deduce

$$I_1 \leq \int_0^R \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} \sigma(B(y,r))\, dy \right)^{\frac{1}{p-1}} \frac{dr}{r^{\frac{n-\alpha p}{p-1}+1}}.$$

Using Fubini's theorem again, we obtain

$$\int_{B(0,R)} \sigma(B(y,r))\, dy \leq \int_{B(0,2R)} |B(y,r)|\, d\sigma = c_n\, r^n\, \sigma(B(0,2R)).$$

Hence, there is a constant $c = c(n,p,\alpha)$ such that

$$I_1 \leq c\, R^{-\frac{n}{p-1}} \sigma(B(0,2R))^{\frac{1}{p-1}} \int_0^R r^{\frac{\alpha p}{p-1}-1} dr$$

$$= c\, R^{\frac{\alpha p - n}{p-1}} \sigma(B(0,2R))^{\frac{1}{p-1}} \leq c \int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Wolff potential estimates

(continuation)

Notice that this is the same estimate we deduced for I_2 with $s = 1$.
Next, we estimate I_1 for $p < 2$ and $s = p - 1$. In this case, we will use the following elementary inequality: for every $R > 0$,

$$\left(\int_0^R \left(\frac{\phi(r)}{r^\gamma} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1} \leq c(p,\gamma) \int_0^{2R} \frac{\phi(r)}{r^\gamma} \frac{dr}{r},$$

where $\gamma > 0$, $1 < p < 2$, and ϕ is a non-decreasing function on $(0, \infty)$.
By this inequality with $\phi(r) = \sigma(B(0, 2r))$ and $\gamma = n - \alpha p$, we obtain:

$$\begin{aligned} I_1 &\leq \frac{c}{|B(0,R)|} \int_{B(0,R)} \int_0^{2R} \frac{\sigma(B(y,r))}{r^{n-\alpha p}} \frac{dr}{r} dy \\ &\leq c\, R^{-n} \sigma(B(0,2R)) \int_0^{2R} r^{\alpha p - 1} dr = c\, R^{-n+\alpha p} \sigma(B(0,2R)) \\ &\leq c \left(\int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1}. \end{aligned}$$

Wolff potential estimates

(continuation)

Combining the estimates for I_1 and I_2, we arrive at

$$\frac{1}{|B(0,R)|}\int_{B(0,R)} (\mathbf{W}_{\alpha,p}\sigma)^s \, dy \le c \left(\int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s .$$

Making the substitution $\rho = 2r$ proves the upper estimate in (18).
To prove the lower estimate of $\mathbf{W}_{\alpha,p}\sigma$, letting $r = 2\rho$ we deduce

$$\mathbf{W}_{\alpha,p}\sigma(y) \ge 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\sigma(B(y,2\rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.$$

For all $y \in B(0,R)$ and $\rho > R$, we have $B(y,2\rho) \supset B(0,\rho)$. Hence,

$$\inf_{B(0,R)} \mathbf{W}_{\alpha,p}\sigma \ge 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\sigma(B(0,\rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}. \qquad \square$$

Wolff potential estimates

(continuation)

Corollary

Suppose $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$.

(i) $\mathbf{W}_{\alpha,p}\sigma \not\equiv +\infty$ *if and only if*

$$\int_1^\infty \left(\frac{\sigma(B(0,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty. \tag{19}$$

(ii) *Condition* (19) *yields*

$$\int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \quad \forall x \in \mathbb{R}^n, \ R > 0. \tag{20}$$

(iii) *If* (19) *holds, then* $\mathbf{W}_{\alpha,p}\sigma \in L^s_{\text{loc}}(dx)$, *where* $s = \min(1, p-1)$, *and*

$$\liminf_{|x|\to\infty} \mathbf{W}_{\alpha,p}\sigma(x) = 0. \tag{21}$$

Wolff's inequality

Wolff's inequality was proved by Th. Wolff using a dyadic model of Wolff's potential. It appeared in [Hedberg-Wolff 1983] in relation to the spectral synthesis problem for Sobolev spaces studied by L. I. Hedberg. Let $\dot{W}^{-\alpha,p'}(\mathbb{R}^n) = \left[\dot{W}_0^{\alpha,p}(\mathbb{R}^n)\right]^*$ denote the dual Sobolev space, where $\frac{1}{p} + \frac{1}{p'} = 1$, $0 < \alpha < \frac{n}{p}$. Define the (α, p)-energy of $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ by

$$\mathcal{E}_{\alpha,p}(\mu) := \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \mu)^{p'} \, dx = \|\mu\|^{p'}_{\dot{W}^{-\alpha,p'}(\mathbb{R}^n)}.$$

Wolff's inequality gives bilateral estimates of $\mathcal{E}_{\alpha,p}(\mu)$ in terms of $\mathbf{W}_{\alpha,p}\mu$.

Theorem (Hedberg-Wolff 1983)

Suppose $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. Then there exists a constant $C = C(\alpha, p, n)$ such that

$$C^{-1}\, \mathcal{E}_{\alpha,p}(\mu) \le \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,p}\mu \, d\mu \le C\, \mathcal{E}_{\alpha,p}(\mu). \tag{22}$$

More general $\mathcal{A}$-Laplace operators

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let us assume that $\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following structural assumptions:

$$x \to \mathcal{A}(x,\xi) \quad \text{is measurable for all } \ \xi \in \mathbb{R}^n,$$

$$\xi \to \mathcal{A}(x,\xi) \quad \text{is continuous for a.e. } \ x \in \Omega,$$

and there are constants $0 < \alpha \le \beta < \infty$, such that for a.e. $x \in \Omega$, and for all ξ in $\mathbb{R}^n$,

$$\mathcal{A}(x,\xi)\cdot\xi \ge \alpha|\xi|^p, \quad |\mathcal{A}(x,\xi)| \le \beta|\xi|^{p-1},$$

$$(\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0 \quad \text{if } \ \xi_1 \neq \xi_2.$$

$\mathcal{A}$-superharmonic solutions

Let $\mu \in \mathcal{M}^+(\Omega)$. We consider the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in } \Omega. \tag{23}$$

A nonlinear potential theory for the equation with measure right-hand side $\mu \in \mathcal{M}^+(\Omega)$,

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu, \tag{24}$$

where u is $\mathcal{A}$-superharmonic, was developed by [Kilpeläinen-Malý '93/94]. They obtained bilateral pointwise estimates of solutions $u \geq 0$ to (24) in terms of Wolff potentials.

Definition. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called $\mathcal{A}$-harmonic if it satisfies the homogeneous equation $\operatorname{div}\mathcal{A}(x, \nabla u) = 0$ in the weak sense. Every $\mathcal{A}$-harmonic function has a continuous representative $\tilde{u} = u$ a.e.

$\mathcal{A}$-superharmonic functions

Next, define $\mathcal{A}$-superharmonic functions via a *comparison principle*:

Definition. A function $u \colon \Omega \to (-\infty, \infty]$ is $\mathcal{A}$-superharmonic if u is *lower semicontinuous*, not identically $+\infty$ in any component of Ω, and, for every open $D \Subset \Omega$ and $h \in C(\overline{D})$, where h is $\mathcal{A}$-harmonic in D,

$$h \le u \text{ on } \partial D \Longrightarrow h \le u \text{ in } D.$$

Some $\mathcal{A}$-superharmonic functions $u \notin \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$. However, for $u \ge 0$, truncates $T_k(u) = \min(u, k) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, $\forall k > 0$. Note that

$$-\mathrm{div}\mathcal{A}(x, \nabla T_k(u)) = \mu_k \ge 0, \quad \mu_k \in \mathcal{M}^+(\Omega),$$

in the weak sense. The generalized gradient $\mathrm{D}u$ of an $\mathcal{A}$-superharmonic function $u \ge 0$ is defined by

$$\mathrm{D}u = \lim_{k \to +\infty} \nabla(T_k(u)).$$

$\mathcal{A}$-superharmonic solutions

Remark. Every $\mathcal{A}$-superharmonic function u has a *quasi-continuous* representative $\tilde{u} = u$ quasi-everywhere (q.e.), that is, everywhere except for a set of p-capacity zero. We assume that u is always chosen this way. Moreover, $u(x) = \liminf_{y \to x} u(y)$ for all $x \in \Omega$.

Let u be $\mathcal{A}$-superharmonic, and let $1 \le r < \frac{n}{n-1}$. Then $|\mathrm{D}u|^{p-1}$, and consequently $\mathcal{A}(x, \mathrm{D}u)$, belongs to $L^r_{\mathrm{loc}}(\Omega)$. This allows us to define a **nonnegative distribution** $-\mathrm{div}\mathcal{A}(x, \mathrm{D}u)$ by

$$-\langle \mathrm{div}\mathcal{A}(x, \mathrm{D}u), \varphi \rangle = \int_\Omega \mathcal{A}(x, \mathrm{D}u) \cdot \nabla \varphi \, dx, \tag{25}$$

for all $\varphi \in C_0^\infty(\Omega)$. Then by the Riesz representation theorem there exists a unique Radon measure $\mu = \mu(u) \in \mathcal{M}^+(\Omega)$ so that

$$-\mathrm{div}\mathcal{A}(x, \mathrm{D}u) = \mu \quad \text{in } \Omega. \tag{26}$$

Renormalized solutions

Consider the equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$ in Ω, where $\mu \in \mathcal{M}^+(\Omega)$, and $\Omega \subseteq \mathbb{R}^n$ is an open set. Let us use the decomposition $\mu = \mu_0 + \mu_s$: μ_0 is absolutely continuous, and μ_s is singular with respect to p-capacity. Let $T_k(s) = \max\{-k, \min\{k, s\}\}$.

Definition. A function $u \in L^{(p-1)s}_{\mathrm{loc}}(\Omega, dx)$ for all $1 \le s < \frac{n}{n-p}$ is called a ***local renormalized solution*** if, for all $k > 0$, $T_k(u) \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\mathrm{D}u \in L^{(p-1)r}_{\mathrm{loc}}(\Omega)$ for all $1 \le r < \frac{n}{n-1}$, and

$$\int_\Omega \langle \mathcal{A}(x, \mathrm{D}u), \mathrm{D}u \rangle \, h'(u) \, \phi \, dx + \int_\Omega \langle \mathcal{A}(x, \mathrm{D}u), \nabla \phi \rangle h(u) \, \phi \, dx$$

$$= \int_\Omega h(u) \, \phi \, d\mu_0 + h(+\infty) \int_\Omega \phi \, d\mu_s,$$

for all $\phi \in C^\infty_0(\Omega)$, and all $h \in W^{1,\infty}(\mathbb{R})$, h' is compactly supported.

$\mathcal{A}$-Laplace operators

Remarks. It is known [Kilpeläinen et al. 2011] that every $\mathcal{A}$-superhamonic solution is a local renormalized solution, and conversely, every local renormalized solution has an $\mathcal{A}$-superharmonic representative.

One can work either with local renormalized solutions, or equivalently with $\mathcal{A}$-superharmonic solutions, or finite energy solutions in the case $u \in W_0^{1,p}(\Omega)$. For finite energy solutions, $\mathbf{D}u$ coincides with the distributional gradient ∇u, and $\mu(u)$ is absolutely continuous with respect to the p-capacity.

Basic facts of potential theory, including nonlinear potential estimates, and the weak continuity principle, hold for the general $\mathcal{A}$-Laplace operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ under the standard structural assumptions imposed above.

Pointwise gradient estimates for $\mathbf{D}u$ and BMO estimates discussed below require some extra assumptions on $\mathcal{A}$.

The Kilpeläinen-Malý theorem (local version)

Theorem (Kilpeläinen-Malý 1994)

Let $\Omega \subset \mathbb{R}^n$ and $B(x, 2R) \subset \Omega$. Let $\mu \in \mathcal{M}^+(\Omega)$ and $1 < p \leq n$. Under the above structural assumptions on $\mathcal{A}$, there exists a constant $C = C(\alpha, \beta, p, n)$ such that

$$\begin{aligned} C^{-1} \mathbf{W}^R_{1,p}\mu(x) &\leq u(x) \\ &\leq C \left[\inf_{B(x,R)} u + \mathbf{W}^{2R}_{1,p}\mu(x) \right], \end{aligned} \tag{27}$$

for any $\mathcal{A}$-superharmonic solution $u \geq 0$ of the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu \quad \text{in } \Omega. \tag{28}$$

Here the truncated Wolff potential of $\mu \in \mathcal{M}^+(\Omega)$ is defined by

$$\mathbf{W}^R_{\alpha,p}\mu(x) := \int_0^R \left(\frac{\mu(B(x,\rho) \cap \Omega)}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \quad x \in \Omega. \tag{29}$$

The Kilpeläinen-Malý theorem (global version)

Corollary (Kilpeläinen-Malý 1994)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ *and* $1 < p < n$. *Under the above structural assumptions on* $\mathcal{A}(x, \xi)$, *there exists a constant* $C = C(\alpha, \beta, p, n)$ *such that*

$$C^{-1}\mathbf{W}_{1,p}\mu(x) \le u(x) \le C\, \mathbf{W}_{1,p}\mu(x), \qquad x \in \mathbb{R}^n, \tag{30}$$

for any p*-superharmonic solution* u *of the equation*

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0. \tag{31}$$

In the case $p \ge n$ *there are no nontrivial solutions to (28) on* $\mathbb{R}^n$.

Moreover, an $\mathcal{A}$-superharmonic solution $u \ge 0$ exists on $\mathbb{R}^n$ if and only if $\mathbf{W}_{1,p}\mu \not\equiv \infty$, that is, condition (17) holds [Phuc-Verbitsky 2008].

Wolff potential estimates
(continuation)

It is easy to see that if $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ is a weak solution to the equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$, then $\mu \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$.

The converse statement is contained in the next lemma.

Lemma

Suppose $1 < p < n$, *and* $\mu \in \mathcal{M}^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. *If* $u \geq 0$ *is an* $\mathcal{A}$*-superharmonic solution to the equation* $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$ *in* $\mathbb{R}^n$, *then* $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n, d\mu)$.

Remark. The proof of the lemma uses Caccioppoli type inequalities and the notion of local renormalized solutions discussed above (see details in [Cao-Verbitsky 2017]).

BMO solutions

It is immediate from pointwise estimates (30) of solutions u to (31) that u is uniformly bounded on $\mathbb{R}^n$ if and only if $\mathbf{W}_{1,p}\mu$ is uniformly bounded.

We next state recent results (joint with Nguyen Cong Phuc) on BMO solutions u to equation (31).

Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is the space of functions u of **bounded mean oscillation** in $\mathbb{R}^n$: $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and there exists a constant C so that

$$\frac{1}{|B|}\int_B |u - \bar{u}_B| dx \le C,$$

for all balls B in $\mathbb{R}^n$, where $\bar{u}_B = \frac{1}{|B|}\int_B u\, dx$.

We will need a class of measures $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ satisfying the Frostman type condition

$$\mu(B(x,R)) \le C R^{n-p}, \quad \forall x \in \mathbb{R}^n,\ R > 0. \tag{32}$$

Notice that $\mathrm{cap}_p(B(x,R)) = c\, R^{n-p}$ where $c = c(p,n)$.

BMO solutions

(continuation)

Theorem 19 (Phuc-Verbitsky 2021)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $1 < p < n$. Then equation (31) has a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ if and only $\mathbf{W}_{1,p}\sigma \not\equiv \infty$ and condition (32) holds, under certain restrictions on $\mathcal{A}$. Moreover, any solution u to (31) lies in $\mathrm{BMO}(\mathbb{R}^n)$ if and only if μ satisfies (32).

Remarks. 1. If μ satisfies (32), then actually any solution u to (31) satisfies the Morrey condition

$$\int_{B(x,R)} |\nabla u|^s dy \le C\, R^{n-s}, \quad \forall x \in \mathbb{R}^n, \ R > 0,$$

provided $1 \le s < p$. This yields $u \in \mathrm{BMO}(\mathbb{R}^n)$ by Poincaré's inequality.
2. The case $p = 2$ of Theorem 13 is due to [D. Adams 1975], and $p \ge 2$ to [G. Mingione 2007] (a local version).

Quasilinear equations with lower order terms

We next consider nontrivial solutions to quasilinear equations of the type

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = \sigma u^q \quad \text{in } \mathbb{R}^n, \tag{33}$$

for $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, under the assumption that the $\mathcal{A}$-Laplace operator of Δ_p type $(1 < p < +\infty)$ obeys the conditions on $\mathcal{A}$ imposed above.
We focus on the sub-natural growth case $0 < q < p-1$. This is an analogue of the sublinear case $0 < q < 1$ for $p = 2$.

We denote by U a positive solution to the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla U) = \sigma, \qquad \liminf_{x\to+\infty} U(x) = 0.$$

Recall that by [Kilpeläinen-Malý 1994],

$$U(x) \approx \mathbf{W}_{1,p}\sigma(x) = \int_0^\infty \left(\frac{\sigma(B(x,\rho))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n.$$

Finite energy solutions

Theorem 20 (Cao-Verbitsky 2016)

Let $1 < p < n$ and $0 < q < p-1$. There exists a solution $u \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ to equation (33) if and only if

$$\int_{\mathbb{R}^n} U^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < +\infty. \tag{34}$$

Moreover, such a solution $u \in L^{1+q}(\mathbb{R}^n, \sigma)$ and is unique. There are no nontrivial solutions on $\mathbb{R}^n$ if $p \geq n$.

Remark. Similar results for inhomogeneous equations $-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu$ hold. A necessary and sufficient condition for $u \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ is given in [Seesanea-V. 2017]:

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < +\infty, \quad \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu) \, d\mu < +\infty.$$

Pointwise estimates of Brezis–Kamin type

Theorem 21 (Cao-Verbitsky 2016)

Let $1 < p < n$ *and* $0 < q < p - 1$. *Let* σ *be a positive measure on* $\mathbb{R}^n$ *such that, for every compact set* $F \subset \mathbf{R}^n$,

$$\sigma(F) \le C \, \mathrm{cap}_p(F).$$

Then there exists a positive solution u *to* (33) *such that* $\liminf_{x \to +\infty} u(x) = 0$, *and*

$$C_1 \, U^{\frac{p-1}{p-1-q}} \le u \le C_2 \left(U + U^{\frac{p-1}{p-1-q}} \right),$$

provided $U \not\equiv +\infty$. *Otherwise there are no nontrivial solutions.*

Remark. For inhomogeneous equations $-\mathrm{div} \mathcal{A}(x, \nabla u) = \sigma u^q + \mu$, similar estimates hold if we add $\mathbf{W}_{1,p}\mu$ to both sides [Verbitsky 2021].

Pointwise estimates in the general case

We consider the weighted norm inequality

$$\|\mathbf{W}_{1,p}\nu\|_{L^q(\Omega,\sigma)} \le \varkappa\, \|\nu\|^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{35}$$

For $B = B(x, r)$, let $\varkappa_B$ be the least constant in the localized inequality

$$\|\mathbf{W}_{1,p}\nu\|_{L^q(\Omega,\sigma_B)} \le \varkappa_B\, \|\nu\|^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n), \tag{36}$$

Then for any nontrivial solution u to (33) we have:

$$u(x) \approx \left(\mathbf{W}_{1,p}\sigma(x)\right)^{\frac{p-1}{p-1-q}} + \int_0^\infty \frac{\left(\varkappa_{B(x,r)}\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\, \frac{dr}{r}. \tag{37}$$

Remark. Similar estimates hold for solutions in the inhomogeneous case $-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu$, with $\mathbf{W}_{1,p}\mu$ on both sides [Verbitsky 2021].

Existence of weak (renormalized) solutions

Theorem 22 (Cao-Verbitsky 2017)

Let $1 < p < n$ and $0 < q < p - 1$. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then there exists a nontrivial (super) solution u to (33) such that $\liminf_{|x|\to+\infty} u(x) = 0$ if and only if the following two conditions hold:

$$\int_1^\infty \left(\frac{\sigma(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \tag{38}$$

$$\int_1^\infty \frac{\left(\varkappa_{B(0,r)}\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \frac{dr}{r} < \infty. \tag{39}$$

In this case any nontrivial solution u satisfies global estimates (37).

Remark. The upper estimate in (37) is proved in [Cao-Verbitsky 2017] for the minimal solution only. True for all solutions [Verbitsky 2021].

Existence of $W^{1,p}_{\text{loc}}$ solutions

If we wish to find a solution u in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, then an additional local version of the condition for finite energy solutions is needed:

$$\int_B \left(\mathbf{W}_{1,p}\sigma_B\right)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma < \infty, \tag{40}$$

for every ball B in $\mathbb{R}^n$.

Theorem 23 (Cao-Verbitsky 2017)

Under the assumptions of the previous theorem, there exists a weak solution $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ *to (33) such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if conditions* (38), (39) *and* (40) *hold. Moreover, global pointwise estimates* (37) *hold for all nontrivial solutions.*

Hessian equations and potential estimates [Trudinger-Wang, 1999; Labutin 2003]

Let F_k ($k = 1, 2, \ldots, n$) be the k-Hessian operator defined by

$$F_k[u] = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{41}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$ on $\mathbb{R}^n$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex in Ω if $F_k[q] \ge 0$ for any quadratic polynomial q such that $u - q$ has a local finite maximum in Ω. A function $u \in C^2_{\text{loc}}(\Omega)$ is k-convex iff

$$F_j[u] \ge 0 \text{ in } \Omega, \quad j = 1, \ldots, k.$$

To a k-convex function u, we associate a k-Hessian measure μ such that $F_k[u] = \mu$ in the viscosity sense. The following pointwise estimates hold [Labutin 2003], [Trudinger-Wang 2002] ([Phuc-Verbitsky 2008] on $\mathbb{R}^n$):

$$u(x) \approx -\mathbf{W}_{\frac{2k}{k+1}, k+1} \mu(x), \quad x \in \mathbb{R}^n.$$

Hessian Equations

Here Wolff's potential is defined by

$$\mathbf{W}_{\frac{2k}{k+1},k+1}\sigma = \int_0^\infty \left(\frac{\sigma(B(x,r))}{r^{n-2k}} \right)^{\frac{1}{k}} \frac{dr}{r}, \quad x \in \mathbb{R}^n,$$

where $k < \frac{n}{2}$. (There are no nontrivial solutions on $\mathbb{R}^n$ if $k \geq \frac{n}{2}$.) Consider the Hessian equation for k-convex functions u such that $\liminf_{|x|\to+\infty} u(x) = 0$:

$$F_k[u] = \sigma\, |u|^q, \quad x \in \mathbb{R}^n, \tag{42}$$

in the sub-natural growth case $0 < q < k$. Then the previous theorems have complete analogues with Wolff's potential $\mathbf{W}_{\frac{2k}{k+1},k+1}$ in place of $\mathbf{W}_{1,p}$ for the p-Laplacian Δ_p.

Hessian equations

Theorem 24 (Cao-Verbitsky 2017)

Let $1 \leq k < \frac{n}{2}$ *and* $0 < q < k$. *Let* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, *and for every compact set* $F \subset \mathbf{R}^n$,

$$\sigma(F) \leq C\, \mathrm{cap}_k(F) \approx \mathrm{Cap}_{\frac{2k}{k+1}, k+1}(F).$$

Then there exists a positive solution u *to (42) such that* $\liminf_{|x|\to+\infty} u(x) = 0$, *and*

$$C_1 (\mathbf{W}_{\frac{2k}{k+1}, k+1}\sigma)^{\frac{k}{k-q}} \leq -u \leq C_2 \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}\sigma + (\mathbf{W}_{\frac{2k}{k+1}, k+1}\sigma)^{\frac{k}{k-q}} \right),$$

provided $\mathbf{W}_{\frac{2k}{k+1}, k+1}\sigma \not\equiv +\infty$. *Otherwise there are no nontrivial solutions.*

Remark. There are complete analogues of the bilateral estimates by means of nonlinear potentials defined in terms of $\varkappa_B$ in the general case.

Proof of bilateral pointwise estimates

We now give a proof of bilateral pointwise estimates [Verbitsky 2021],

$$u(x) \approx (\mathrm{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \int_0^\infty \frac{(\varkappa(B(x,r)))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \frac{dr}{r} \qquad (43)$$
$$+ \mathrm{W}_{1,p}\mu(x),$$

for all nontrivial solutions of the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0, \qquad (44)$$

in the case $0 < q < p-1$, where $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$.

Remark. A proof of the lower estimate for **all** solutions, along with the upper estimate in (43) in the case $\mu = 0$ for the minimal solution only was provided in [Cao-Verbitsky 2017].

Proof of bilateral pointwise estimates

Homogeneous equations

We first consider the case $\mu = 0$, that is, nontrivial solutions to the homogeneous equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma u^q \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0. \tag{45}$$

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p-1$. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. For simplicity, the Wolff potential $\mathbf{W}_{\alpha,p}\sigma$ will be denoted by $\mathbf{W}\sigma$, i.e.,

$$\mathbf{W}\sigma(x) = \int_0^\infty \left[\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n. \tag{46}$$

We denote by $\varkappa$ the least constant in the weighted norm inequality

$$\|\mathbf{W}\nu\|_{L^q(\mathbb{R}^n, d\sigma)} \le \varkappa\, \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{47}$$

Intrinsic potentials

Remark. It is easy to see using the Kilpeläinen-Malý theorem that the embedding constant $\varkappa$ in the case $\alpha = 1$ is equivalent to the constant κ in the inequality

$$\|\phi\|_{L^q(\mathbb{R}^n, d\sigma)} \le \kappa \, \|\mathrm{div}\mathcal{A}(x, \nabla\phi)\|^{\frac{1}{p-1}}, \tag{48}$$

for all $\mathcal{A}$-superharmonic $\phi \ge 0$ which vanish at ∞.

We will need a localized version of inequality (47) for $\sigma_B = \sigma|_B$, where B is is a ball in $\mathbb{R}^n$, and $\varkappa(B)$ is the least constant in

$$\|\mathbf{W}\nu\|_{L^q(\mathbb{R}^n, d\sigma_B)} \le \varkappa(B) \, \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{49}$$

The intrinsic potential of Wolff type $\mathbf{K}\sigma = \mathbf{K}_{\alpha,p,q}\sigma$ is defined in terms of $\varkappa(B(x,t))$, the least constant in (49) with $B = B(x,t)$:

$$\mathbf{K}\sigma(x) = \int_0^\infty \left[\frac{\varkappa(B(x,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n. \tag{50}$$

Intrinsic potentials

It is easy to see that $\mathbf{K}\sigma \not\equiv \infty$ if and only if

$$\int_a^\infty \left[\frac{\varkappa(B(0,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} < \infty, \tag{51}$$

for any (equivalently, all) $a > 0$, provided $\varkappa(B) < \infty$ for all balls B. This is similar to the condition $\mathbf{W}\sigma \not\equiv \infty$, which is equivalent to

$$\int_a^\infty \left[\frac{\sigma(B(0,t))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} < \infty. \tag{52}$$

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p-1$. Let us fix $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. We start with some estimates of solutions to the equation

$$u(x) = \mathbf{W}(u^q d\sigma)(x), \quad u \geq 0, \quad x \in \mathbb{R}^n. \tag{53}$$

Integral equations with Wolff potentials

Remarks. In equation (53), $u < \infty$ $d\sigma$-a.e. (or equivalently $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \sigma)$), and also (53) is understood $d\sigma$-a.e.

In this case, we can choose a representative $\tilde{u}$ such that $\tilde{u} = u$ $d\sigma$-a.e., defined for all $x \in \mathbb{R}^n$ by $\tilde{u}(x) := \mathbf{W}(u^q d\sigma)(x)$. Then clearly $\tilde{u}(x) = \mathbf{W}(\tilde{u}^q d\sigma)(x)$ for all $x \in \mathbb{R}^n$, and $\tilde{u}$ is a solution to (53) defined everywhere on $\mathbb{R}^n$.

We will always use such representatives, denoted simply by u, so that (53) is considered everywhere. Our goal is to give bilateral pointwise estimates of solutions to $u(x) = \mathbf{W}(u^q d\sigma)(x)$ for all $x \in \mathbb{R}^n$ where $u(x) < \infty$.

We also treat the corresponding **subsolutions** $u \geq 0$ such that

$$u(x) \leq \mathbf{W}(u^q d\sigma)(x) < \infty, \quad x \in \mathbb{R}^n, \tag{54}$$

and **supersolutions** $u \geq 0$ such that

$$\mathbf{W}(u^q d\sigma)(x) \leq u(x) < \infty, \quad x \in \mathbb{R}^n, \tag{55}$$

considered $d\sigma$-a.e., and at every $x \in \mathbb{R}^n$ where these inequalities hold.

Integral equations with Wolff potentials

For any $\nu \in \mathcal{M}^+(\mathbb{R}^n)$ $(\nu \neq 0)$ such that $\mathbf{W}\nu \not\equiv \infty$, we set

$$\phi_\nu(x) := \mathbf{W}\nu(x) \left(\frac{\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x)}{\mathbf{W}\nu(x)} \right)^{\frac{p-1}{p-1-q}}, \quad x \in \mathbb{R}^n, \tag{56}$$

where we assume that $\mathbf{W}\nu(x) < \infty$.
Next, for $x \in \mathbb{R}^n$, we set

$$\phi(x) := \sup\{\phi_\nu(x) : \ \nu \in \mathcal{M}^+(\mathbb{R}^n), \nu \neq 0, \mathbf{W}\nu(x) < \infty\}. \tag{57}$$

Theorem 25

Any nontrivial solution $u \geq 0$ *to* (53) *satisfies the estimates*

$$C\,\phi(x) \leq u(x) \leq \phi(x), \qquad x \in \mathbb{R}^n, \tag{58}$$

where C *is a positive constant which depends only on* p, q, α *and* n. *Moreover, the upper bound in* (58) *holds for any* **subsolution** u, *whereas the lower bound in* (58) *holds for any nontrivial* **supersolution** u.

Proof of Theorem 25

The proof of Theorem 25 is based on a series of lemmas.

Lemma 1

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $0 < q < p - 1$. *Let* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Suppose* u *is a subsolution to* (53). *Then*

$$u(x) \le \phi(x), \qquad x \in \mathbb{R}^n, \tag{59}$$

provided $\mathbf{W}(u^q d\sigma)(x) < \infty$. *In paticular,* (59) *holds* $d\sigma$*-a.e.*

Proof. Setting $d\nu = u^q d\sigma$, we see that $u(x) \le \mathbf{W}\nu(x) < \infty$, and consequently $\mathbf{W}\nu(x) \le \mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x)$. Then clearly,

$$\phi_\nu(x) := \mathbf{W}\nu(x) \left(\frac{\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x)}{\mathbf{W}\nu(x)} \right)^{\frac{p-1}{p-1-q}} \ge \mathbf{W}\nu(x).$$

Proof of Theorem 25

Hence,

$$u(x) \le \phi_\nu(x), \quad x \in \mathbb{R}^n,$$

which yields immediately (59). □

Lemma 2

Let $\nu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then there exists a positive constant C which depends only on p, q, α, and n such that

$$\begin{aligned} \mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x) \le C\,(\mathbf{W}\nu(x))^{\frac{q}{p-1}} \\ \times \left[\mathbf{W}\sigma(x) + (\mathbf{K}\sigma(x))^{\frac{p-1-q}{p-1}}\right], \quad x \in \mathbb{R}^n. \end{aligned} \tag{60}$$

Proof. Without loss of generality we may assume that $\nu \neq 0$ and $\mathbf{W}\nu(x) < \infty$. For $x \in \mathbb{R}^n$, we have

$$\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x) = \int_0^\infty \left[\frac{\int_{B(x,t)} (\mathbf{W}\nu(y))^q d\sigma(y)}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}. \tag{61}$$

Proof of Theorem 25

For $y \in B(x,t)$, we have that $B(y,r) \subset B(x,2t)$ if $0 < r \le t$, and $B(y,r) \subset B(x,2r)$ if $r > t$. Consequently, for $y \in B(x,t)$,

$$\mathbf{W}\nu(y) = \int_0^t \left[\frac{\nu(B(y,r))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} + \int_t^\infty \left[\frac{\nu(B(y,r))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}$$

$$\le \int_0^t \left[\frac{\nu(B(y,r)\cap B(x,2t))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} + \int_t^\infty \left[\frac{\nu(B(x,2r))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}$$

$$\le \mathbf{W}\nu_{B(x,2t)}(y) + c\,\mathbf{W}\nu(x), \quad \text{where } c = 2^{\frac{n-\alpha p}{p-1}}. \text{ Hence,}$$

$$\int_{B(x,t)} (\mathbf{W}\nu(y))^q d\sigma(y) \le \int_{B(x,t)} \left(\mathbf{W}\nu_{B(x,2t)}(y)\right)^q d\sigma(y) + c^q \, (\mathbf{W}\nu(x))^q \, \sigma(B(x,t)).$$

Proof of Theorem 25

Notice that by (49),

$$\int_{B(x,t)} \left(\mathbf{W}\nu_{B(x,2t)}(y)\right)^q \, d\sigma(y) \leq \kappa(B(x,t))^q \, \nu(B(x,2t))^{\frac{q}{p-1}}.$$

Combining the preceding estimates, we deduce

$$\int_{B(x,t)} (\mathbf{W}\nu(y))^q d\sigma(y) \leq \kappa(B(x,t))^q \, \nu(B(x,2t))^{\frac{q}{p-1}} + c^q \, (\mathbf{W}\nu(x))^q \, \sigma(B(x,t)).$$

It follows from (61) and the preceding estimate,

$$\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x) \leq c \int_0^\infty \left[\frac{\kappa(B(x,t))^q \, \nu(B(x,2t))^{\frac{q}{p-1}}}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} + c \, (\mathbf{W}\nu(x))^{\frac{q}{p-1}} \int_0^\infty \left[\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} = c\,(I + II).$$

Proof of Theorem 25

By Hölder's inequality with exponents $\frac{p-1}{p-1-q}$ and $\frac{p-1}{q}$, we estimate

$$I = \int_0^\infty \left[\frac{\kappa(B(x,t))^q \, \nu(B(x,2t))^{\frac{q}{p-1}}}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}$$

$$\leq \left(\int_0^\infty \left[\frac{\nu(B(x,2t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right)^{\frac{q}{p-1}}$$

$$\times \left(\int_0^\infty \left[\frac{\kappa(B(x,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right)^{\frac{p-1-q}{p-1}}$$

$$= 2^{\frac{q(n-\alpha p)}{(p-1)^2}} \, (\mathbf{W}\nu(x))^{\frac{q}{p-1}} \, (\mathbf{K}\sigma(x))^{\frac{p-1-q}{p-1}} \, .$$

Proof of Theorem 25

Clearly,

$$II = (\mathbf{W}\nu(x))^{\frac{q}{p-1}} \int_0^\infty \left[\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} = (\mathbf{W}\nu(x))^{\frac{q}{p-1}} \, \mathbf{W}\sigma(x).$$

We deduce

$$\begin{aligned} &\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x) \le c(I + II) \\ &\le c\, (\mathbf{W}\nu(x))^{\frac{q}{p-1}} \left[\mathbf{W}\sigma(x) + (\mathbf{K}\sigma(x))^{\frac{p-1-q}{p-1}}\right]. \end{aligned}$$

This completes the proof of (60). □

Proof of Theorem 25

Lemma 3

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $0 < q < p-1$. *Let* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Then there exist positive constants* C_1, C_2 *which depend only on* p, q, α *and* n *such that*

$$C_1\, \phi(x) \le (\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \le C_2\, \phi(x), \qquad (62)$$

where the lower estimate holds for all $x \in \mathbb{R}^n$, *whereas the upper estimate holds provided* $\mathbf{W}\sigma(x) < \infty$ *and* $\mathbf{K}\sigma(x) < \infty$.
If $\mathbf{W}\sigma \not\equiv \infty$ *and* $\mathbf{K}\sigma \not\equiv \infty$, *then* $\phi < \infty$ $d\sigma$*-a.e., and the upper estimate in* (62) *holds* $d\sigma$*-a.e.*

Proof of Theorem 25

Proof. To prove the upper estimate in (62), notice that, if $\mathbf{W}\sigma \not\equiv \infty$ and $\mathbf{K}\sigma \not\equiv \infty$, it follows from [Cao-V. 2017], Theorem 4.8, that there exists a (minimal) solution u to (53) such that

$$\begin{aligned} c_1 \, \Big[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \Big] &\le u(x) \\ &\le c_2 \, \Big[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \Big], \quad x \in \mathbb{R}^n, \end{aligned} \tag{63}$$

where c_1, c_2 are positive constants which depend only on p, q, α and n. The lower bound in (63) holds for any nontrivial supersolution u as was shown in [Cao-V. 2017], Theorems 4.8, $d\sigma$-a.e., and in fact at every $x \in \mathbb{R}^n$ where $\mathbf{W}(u^q d\sigma)(x) \le u(x)$, as is clear from the proof.
For the **minimal** solution u, we have $u(x) = \mathbf{W}(u^q d\sigma)(x) < \infty$, provided $\mathbf{W}\sigma(x) < \infty$ and $\mathbf{K}\sigma(x) < \infty$, by the upper estimate in (63).

Proof of Theorem 25

Thus, by Lemma 1 and the lower bound in (63), we deduce

$$c_1 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right] \le u(x) \le \phi(x).$$

If $\mathbf{W}\sigma \not\equiv \infty$ and $\mathbf{K}\sigma \not\equiv \infty$, then as indicated above, there exists a solution u to (53) such that $u = \mathbf{W}(u^q d\sigma) < \infty$ $d\sigma$-a.e., and (63) holds $d\sigma$-a.e. It follows that $\mathbf{W}\sigma < \infty$ and $\mathbf{K}\sigma < \infty$ $d\sigma$-a.e., and hence $\phi < \infty$ $d\sigma$-a.e. by the lower estimate in (62) (Lemma 2). Letting $d\nu = u^q d\sigma$, we deduce $u \le \phi_\nu \le \phi$ $d\sigma$-a.e., so that (62) holds $d\sigma$-a.e. as well. The proof of Lemma 3 is complete. □

Proof of Theorem 25. The upper bound in (58) for any subsolution u follows from Lemma 1, whereas the lower bound for any nontrivial supersolution u follows from the lower bound in (63) and Lemma 3:

$$u(x) \ge c_1 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right] \ge c_1\, C_1\, \phi(x), \quad x \in \mathbb{R}^n.$$